

Lens space determinants

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More analysis of operator determinants on homogeneous three dimensional lens spaces is presented with the emphasis on numerics so that Laplacians for massive fields can be dealt with. Polyhedral quotients are also briefly considered. Twisted fields, corresponding to flat connections, are looked at and examples of determinants computed. Twisted cyclic quantities are sufficient to determine those for any twisting on any factor.

An application to the thermodynamics on sphere quotients is given. Some computations are made for inhomogeneous spaces and higher dimensions are commented on. Minimal coupling is also dealt with.

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1. Introduction

In [1,2] I computed the operator determinants on lens space factors of the unit 3-sphere for the Laplacian conformal in four dimensions. This meant that the eigenvalues were squares of integers and the ζ -function analysis was relatively straightforward and explicit. Nash and O'Connor, [3], evaluated the determinants for other operators in connection with analytic torsion. Factors of higher spheres have been considered by Bauer and Furutani [4]. Evaluations have also been conducted lately in string theory and quantum field theory, *e.g.* Castro, Lashkar and Maloney, [5], Gang, [6], Alday, Fluder and Sparks, [7], Radićević, [8].

I here return to this topic still restricting to the 3-sphere, essentially because it is somewhat special and therefore more interesting (and easier). My attitude is mainly numerical and I wish to generalise the Laplacian to one that includes mass, in particular to the Laplacian conformal on the three-sphere. The analysis may be useful when considering the topology of the Universe, a topic under present scrutiny. I feel that a mostly computational approach has advantages such as speed and will prove a useful adjunct to the analytical route.

Fixed-point-free factors of the three sphere, S^3/Γ' , divide into the homogeneous variety in which the deck group, Γ' , acts on the right (or left) the left (or right) action being trivial, and the inhomogeneous sort with a two-sided action. Left and right here refer to the $SU(2)$ groups in the symmetry group isomorphism $SO(4) \sim SU(2) \times SU(2)/\mathbb{Z}_2$ acting on S^3 in its guise as $SU(2)$, *i.e.* $\Gamma' = \Gamma'_L \times \Gamma'_R$.

In this paper I mostly confine my attention to the homogeneous type which is somewhat easier to deal with. The classification of relevance here is given in Wolf, [9], Corollary 2.7.2 which says that Γ'_R is either a cyclic group or a binary polyhedral group ².

I showed in [1] that, because of the homomorphism $SO(3) \sim SU(2)/\mathbb{Z}_2$, the binary degeneracies, which are the essential part of the spectrum, can be reduced to an analysis on the *orbifolded* two-sphere, S^2/Γ , where Γ' is the lift of Γ . Following Bethe, I refer to Γ' as the *double* of Γ . For many purposes, one does not need the degeneracies themselves, their generating functions often being sufficient, and these are classically available, *e.g.* [11].

Lens spaces arise from the choice of a cyclic group, \mathbb{Z}_q , for Γ'_R . In order to use the reduction mentioned in the last paragraph, the restriction that q should be even

² I consider the dihedron to be a polyhedron. Incidentally, my favourite potted summary of the classification is that by Ellis, [10].

has to be made because the binary lift of \mathbb{Z}_q is \mathbb{Z}_{2q} . However it is possible to treat even and odd orders together, which I will do.

2. Massive determinants on homogeneous lens spaces

In order to be in accord with a previous notation, the scalar eigenvalues are taken to be $l^2 - \alpha^2$, $l = 1, 2, \dots$, with *total* degeneracies, $D_l(\Gamma')$. I need the associated ζ -function which I write using a Bessel function relation, [12,13], as, *cf* [14],

$$\begin{aligned} Z(s, \Gamma', \alpha) &= \sum_l \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^s} \\ &= \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty d\tau \sum_l D_l(\Gamma') e^{-l\tau} \left(\frac{\tau}{2\alpha} \right)^{s-1/2} I_{s-1/2}(\alpha\tau) \\ &= \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty d\tau K_{\Gamma'}^{1/2}(\tau) \left(\frac{\tau}{2\alpha} \right)^{s-1/2} I_{s-1/2}(\alpha\tau), \end{aligned} \quad (1)$$

where $K_{\Gamma'}^{1/2}$ is the cylinder kernel for the Laplacian conformal in four dimensions. and is, equivalently, the degeneracy generating function used in [1].

For the homogeneous case, the total degeneracy is a product of the left and right degeneracies, the former simply being a factor of l . (One should regard l as an $SU(2)$ representation dimension, $2j + 1$, which is just the range of the left index on a Wigner $\mathcal{D}^{(j)}$ matrix.) Then, $D_l(\Gamma') = l d_l(\Gamma')$ and we need the second factor.

For quotient spaces, simple character theory allows a closed form for the generating function of right degeneracies. In particular, for lens space degeneracies, $d_l(q)$,

$$G(t, q) \equiv \sum_{l=1}^{\infty} d_l(q) t^l = \frac{t(1 + t^q)}{(1 - t^2)(1 - t^q)}, \quad (2)$$

and, therefore, setting $t = e^{-\tau}$,

$$\begin{aligned} K_q^{1/2}(\tau) &= \sum_{l=1}^{\infty} l d_l(q) e^{-l\tau} \\ &= -\frac{d}{d\tau} \frac{\coth(q\tau/2)}{2 \sinh \tau}. \end{aligned} \quad (3)$$

The mathematical problem before us is the evaluation of the derivative at zero, $Z'(0, q, \alpha)$, of the ζ -function (1) with (3). Since the dimension of the manifold is odd, I can proceed exactly as in [13], [14], [15] and employ a complex contour method of continuing to $s = 0$.

Defining $I(s, q, \alpha)$ by $Z(s, q, \alpha) = I(s, q, \alpha)/\Gamma(s)$, one has $Z'(0, q, \alpha) = I(0, q, \alpha)$ and a simple calculation which parallels that in [14] produces,

$$I(0, q, \alpha) = \int_0^\infty dx \operatorname{Re} \frac{(q \sinh \tau + \cosh \tau \sinh q\tau) \cosh \alpha\tau}{2\tau \sinh^2 \tau \sinh^2 q\tau/2}, \quad (4)$$

where $\tau = x + i\Delta$ with $\Delta < 2\pi/q$. For S^3 , $q = 1$ and,

$$\begin{aligned} I(0, 1, \alpha) &= \int_0^\infty dx \operatorname{Re} \frac{(1 + \cosh \tau) \cosh \alpha\tau}{2\tau \sinh \tau \sinh^2 \tau/2} \\ &= \int_0^\infty dx \operatorname{Re} \frac{\coth \tau/2 \cosh \alpha\tau}{2\tau \sinh^2 \tau/2}, \end{aligned}$$

which, as a check, agrees with an expression derived in [14].

In Fig.1, I plot $\log \det = -I(0, q, \alpha)$ as a continuous function of q for $\alpha = 0$ and for $\alpha = 1/2$, corresponding to conformal in four and in three dimensions respectively.

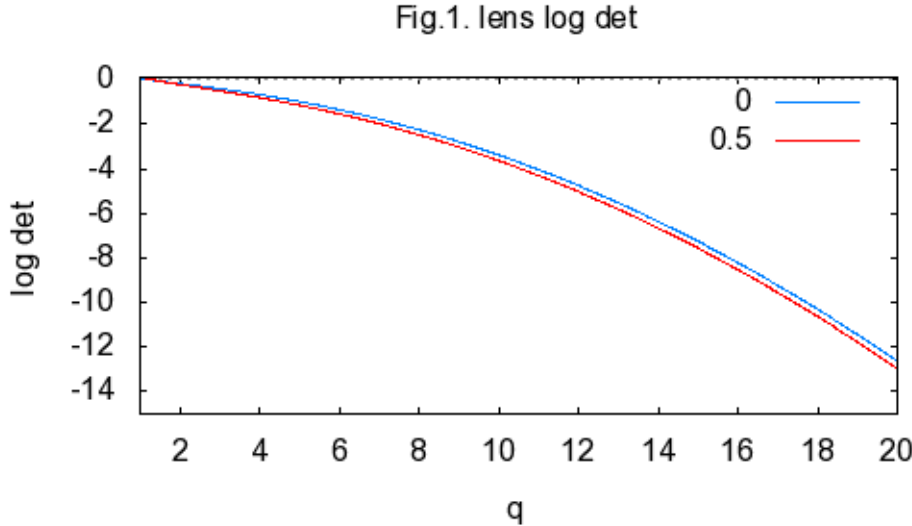
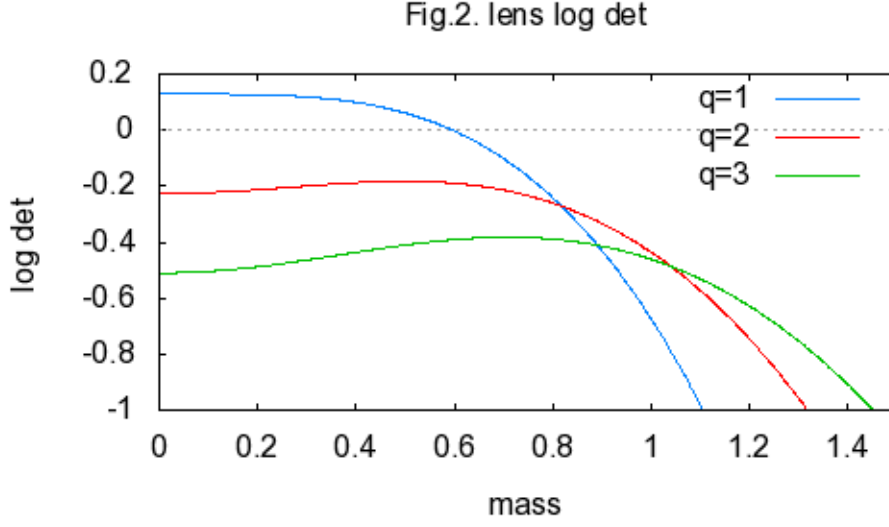


Figure 2 gives, for three values of q , the variation of $\log \det$ against the “mass”, μ defined by $\mu^2 = 1/4 - \alpha^2$, which is, for this purpose, a more convenient variable. (What should be called the mass is somewhat arbitrary.)

The asymptotic variation for large mass was discussed, in a similar context, in [14] and is best expressed in terms of the short-time expansion coefficients of the heat-kernel for propagation by the Laplacian conformal in *four* dimensions because this expansion, in the present case, terminates with the first, ‘volume’ term and it easily follows that $Z'(0, q, im) \sim \pi m^3/3q$ with an exponentially small remainder. The beginnings of this can be seen in the figure.



3. The other quotients

As a simple ‘application’ of these expressions, the logdets for the binary tetrahedral, (\mathbf{T}') , octahedral, (\mathbf{O}') , and icosahedral, (\mathbf{I}') , factored spaces can be determined using the cyclic decomposition of the corresponding $\text{SO}(3)$ groups, [11,16]. This was used in [1]. Any spectral quantity³ $\mathcal{S}(\Gamma')$ on such an S^3/Γ' can be expressed in terms of binary cyclic quantities,

$$\mathcal{S}(\Gamma') = \frac{1}{2} \left(\sum_q \mathcal{S}(\mathbb{Z}_{2q}) - \mathcal{S}(\mathbb{Z}_2) \right), \quad (5)$$

where the sum is over the (three) q -fold axes of rotation in the $\text{SO}(3)$ subgroup, Γ , of which Γ' is the lift. The values of q are contained in the symbol of the polyhedral group, (l, m, n) , *i.e.* $l = 2, m = 3$ and $n = 3, 4, 5$ for $\mathbf{T}, \mathbf{O}, \mathbf{I}$, respectively.

I give the numbers for the conformal-in-three and conformal-in-four dimensions Laplacians. The latter agree with those in [1] computed using a different algorithm for the even lens spaces. Thus,

$$\det(\mathbf{T}') = 0.159259, \quad \det(\mathbf{O}') = 0.099650, \quad \det(\mathbf{I}') = 0.055743,$$

³ I restrict the notion of spectral quantity by imposing linearity in the sense that, if the spectrum is composed of the union of two sets, its spectral quantity is the sum of the spectral quantities of the two parts. A typical example is the ζ -function but not the determinant.

for conformal-in-three, and,

$$\det(\mathbf{T}') = 0.202089, \quad \det(\mathbf{O}') = 0.128776, \quad \det(\mathbf{I}') = 0.073056,$$

for conformal-in-four dimensions. These values are for real scalar fields.

4. Twisted fields

A simple, and basic, example is a complex scalar field or U(1) line bundle. In general terms, the quantum field theory is characterised by the homomorphisms of the deck group, Γ' , which here is the fundamental group, into the bundle group, \mathcal{G} , *i.e.* by $\text{Hom}(\Gamma', \mathcal{G})$. which can be realised as flat connections associated with ‘fluxes’ of gauge fields through ‘holes’ in the configuration space, *e.g.* [17–20].

For the simplest situation, $\Gamma' = \mathbb{Z}_q$ and $\mathcal{G} = \text{U}(1)$, as considered in [17], for example. The homomorphisms correspond to the q q th roots of unity and the elements of $\text{Hom}(\mathbb{Z}_q, \text{U}(1))$, *i.e.* the multiplying representations of \mathbb{Z}_q (the ‘twists’) are generated by ω^r where $\omega^q = 1$ and $0 \leq r < q$.

Instead of (2), character theory gives the degeneracy generating function

$$G(t, q, r) \equiv \sum_{l=1}^{\infty} d_l(q, r) t^l = \frac{t^{1+r}(1+t^{-2q\delta})}{(1-t^2)(1-t^q)},$$

where $\delta + 1/2 = r/q$, which is the flux. For (3) one has,

$$\begin{aligned} K_q^{1/2}(\tau, r) &= \sum_{l=1}^{\infty} l d_l(q, r) e^{-l\tau} \\ &= -\frac{d}{d\tau} \frac{\cosh(q\tau\delta)}{2 \sinh \tau \sinh q\tau/2}. \end{aligned} \tag{6}$$

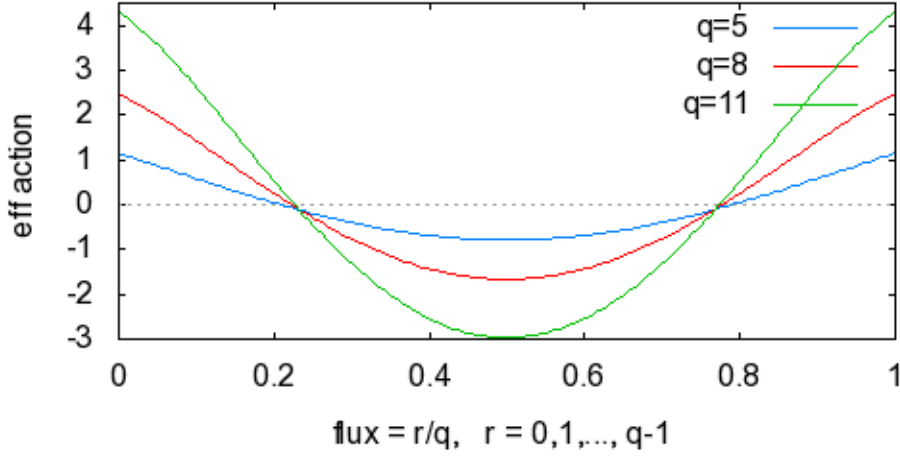
The effect of the twisting on the logdet can now be determined numerically from the equivalent of (4), *i.e.* , $(\tau = x + i\Delta)$,

$$I(0, q, r, \alpha) = - \int_0^{\infty} dx \text{Re} \left(\frac{\cosh \alpha \tau}{\tau} \frac{d}{dx} \frac{\cosh(q\tau\delta)}{2 \sinh \tau \sinh q\tau/2} \right), \tag{7}$$

where I have not, this time, performed the differentiation.

Figure 3 shows the (typical) variation of the effective action (minus half the logdet), for a complex field with the twist of the bundle plotted continuously. The mass parameter, α , has been set to 1/2 to give the Laplacian conformal in 3-d.

Fig.3. twisted lens effective action



5. More complicated twistings

More generally, if the field belongs to some representation of the internal symmetry (bundle) group, \mathcal{G} , a twisting, *i.e.* an element $\rho \in \text{Hom}(\Gamma', \mathcal{G})$ is determined by expressing this representation as a direct sum of irreps, A , of Γ' . All that is needed, for a flat bundle, is that the dimensions match and that any other conditions, such as unimodularity, be met, [21]. This is an enumerative, rather than an algorithmic, prescription for cataloguing the gauge vacua.

For a flat bundle spectral quantity, $\mathcal{S}(\rho)$, the direct sum becomes, by additivity, an algebraic sum of the components, $\mathcal{S}(A)$, which act as spectral building blocks. These, in turn can be expressed linearly in terms of *cyclic* spectral quantities. This follows from the algebraic sufficiency of the cyclic subgroups of Γ' , which is guaranteed by Artin's theorem, and an inducing isospectral theorem of Ray and Singer, [22]. (See [23], Tsuchiya, [24].) Equation (5) is an example which I rewrite more explicitly as,

$$\mathcal{S}(1) = \frac{1}{2}(\mathcal{S}(0; R) + \mathcal{S}(0; S) + \mathcal{S}(0; T) - \mathcal{S}(0; RST)), \quad (8)$$

where $\mathcal{S}(r; \gamma)$ is the twisted spectral quantity for a cyclic quotient generated by γ . R, S and T are the generators for the binary polyhedral groups, *e.g.* [25], corresponding to $q = 2$, $q = 3$ and $q = (3, 4 \text{ or } 5)$, respectively.⁴ The left-hand side of (8) refers to the trivial irrep, 1, of Γ' , *i.e.* the trivial bundle.

⁴ Only R, S and T are really needed. The inclusion of the central element $\overline{E} = RST$ gives a more symmetrical formulation, $\overline{E}^2 = E = \text{id}$.

As another example, derived in [23], I give the first twisted relation,

$$\mathcal{S}(2_s) = \frac{1}{2}(\mathcal{S}(1; R) + \mathcal{S}(1; S) + \mathcal{S}(1; T) - \mathcal{S}(1; RST)), \quad (9)$$

where 2_s is the two-dimensional spinor representation of \mathbf{T}' , \mathbf{O}' or \mathbf{I}' .

Numerical evaluation of (9) using (7) produces,

$$\det(\mathbf{T}') = 0.652112, \quad \det(\mathbf{O}') = 0.439366, \quad \det(\mathbf{I}') = 0.260126,$$

for conformal in three, and,

$$\det(\mathbf{T}') = 0.663348, \quad \det(\mathbf{O}') = 0.454594, \quad \det(\mathbf{I}') = 0.272797,$$

for conformal in four dimensions.

Just to illustrate the sort of calculation that is possible, I consider a (scalar) field belonging to the fundamental rep of an internal $U(3)$ on icosahedral space, S^3/\mathbf{I}' . There are four elements of $\text{Hom}(\mathbf{I}', U(3))$, *viz* 3 , $3'$, $1 \oplus 2_s$, $1 \oplus 2'_s$ and the trivial one, $1 \oplus 1 \oplus 1$.⁵ Inducing representations shows that, [23],

$$\begin{aligned} \mathcal{S}(2'_s) &= \mathcal{S}(1; S) - \frac{1}{2}\mathcal{S}(5; T) - \mathcal{S}(1; T) \\ \mathcal{S}(3') &= \frac{1}{2}\mathcal{S}(2; R) - \mathcal{S}(2; T) \\ \mathcal{S}(3) &= \frac{1}{2}\mathcal{S}(2; R) - \mathcal{S}(4; T). \end{aligned}$$

Together with (8) and (9) these yield the following numbers for the 4d-conformal Laplacian,

$$\begin{aligned} \det(1 \oplus 1 \oplus 1) &= 0.000391 \\ \det(1 \oplus 2_s) &= 0.019929 \\ \det(1 \oplus 2'_s) &= 0.021993 \\ \det(3) &= 0.164545 \\ \det(3') &= 2.00091 \end{aligned}$$

A crude physical conclusion might be that, if the smallest effective action ($-\log \det/2$) is preferred, on the basis of some unspecified dynamics, then the $3'$ twisting stands out as the lowest, for this artificial situation. The symmetry is broken from $U(3)$ to $U(1)$.

⁵ The irreps are labeled by their dimensions. The subscript s stands for ‘spinor’.

6. Application to thermodynamics on factored spaces

In [21] a related analysis was made of Witten's symmetry breaking mechanism induced, as a toy model, by fluxes in geometries involving factors of the three-sphere. The Casimir energies were evaluated in the different gauge vacua and closed forms were found for lens spaces, the fields being in the adjoint representation of $\mathcal{G} = \text{SU}(n)$. The space-time was a factored Einstein universe, $T \times \text{S}^3/\Gamma'$. In [26,27], the system was put at a finite temperature, and symmetry restoration for increasing temperature analysed. The object of present interest, the effective action on S^3/Γ' , enters into this scheme through the high temperature expansions of thermodynamic quantities such as the free energy.

General asymptotic series are given in [28] in terms of the coefficients of the short-time expansion of the heat-kernel, apart from one term which stems from the zero mode on the thermal circle. To a constant factor, this term is $Z'(0, \Gamma', \alpha)$. Furthermore, for the case of conformal propagation in four-dimensions, and scalar fields, the heat-kernel expansion terminates with the first volume, or Weyl, term, on fixed-point-free factors of the three-sphere, as already stated. The high temperature expansions, [28], of the, here finite, free energy, internal energy and entropy then simplify drastically to, (for a real scalar),

$$\begin{aligned} F &\sim -\frac{\pi^4}{45|\Gamma'|}\beta^{-4} - \frac{1}{2} Z'(0, \Gamma', 0) \beta^{-1} \\ E &\sim \frac{\pi^4}{15|\Gamma'|}\beta^{-4} \\ S &\sim \frac{4\pi^4}{45|\Gamma'|}\beta^{-3} + \frac{1}{2} Z'(0, \Gamma', 0), \end{aligned} \tag{10}$$

respectively, with the error being exponentially small.

These expansions have been rederived recently by Asorey *et al*, [29]. This reference also contains expressions for the complete thermodynamic quantities.

Thermal theory on factors of the three-sphere had been set up by Kennedy, *passim*, [30], and was pursued in more detail by Unwin, [31].

As noted in [32] and [1], knowledge of the degeneracy generating functions allows one to write down the free energy, say, with minimum fuss, in terms of the cylinder kernel, $K^{1/2}$, as,

$$F(\beta) = E_0 - \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} K^{1/2}(m\beta),$$

where E_0 is the zero temperature internal energy or vacuum energy, or Casimir energy.

In particular for twisted q -lens spaces, using (6),

$$F(\beta, q, \delta) = E_0(q, \delta) + \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{d}{d\beta} \frac{\cosh(qm\beta\delta)}{\sinh m\beta \sinh qm\beta/2}, \quad (11)$$

which is a way of rewriting the standard statistical sum over states.

The twisted Casimir energies are effectively available in [1]. Equation (6) for the twisted cylinder kernel should be compared with equation (46) in [1]. The integer ‘degrees’ there, here transcribe to the reals,

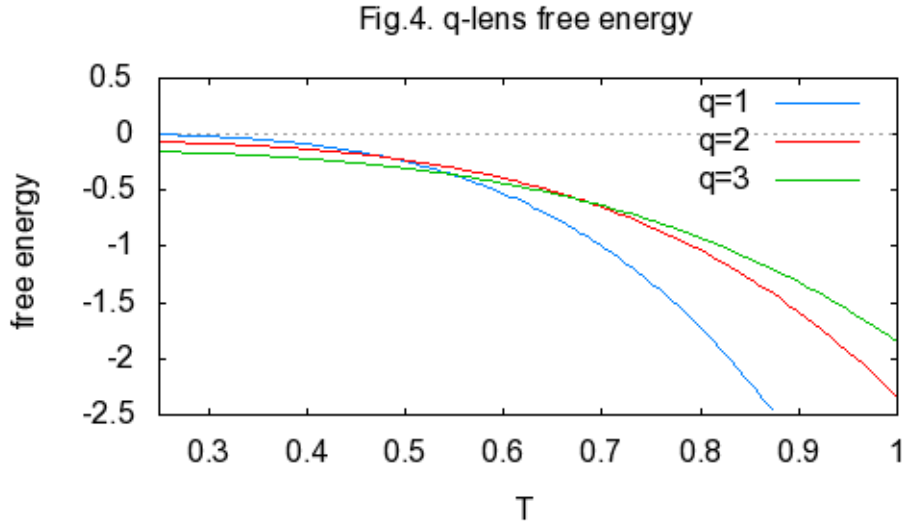
$$\delta_0 = q\delta, \quad \delta_1 = q/2, \quad \delta_2 = 1,$$

and the Casimir energies are given in [1] equn.(52) ⁶. Thus,

$$E_0(q, \delta) = \frac{(7 - 120\delta^2 + 240\delta^4)q^4 + 40(1 - 12\delta^2)q^2 + 112}{2880q}. \quad (12)$$

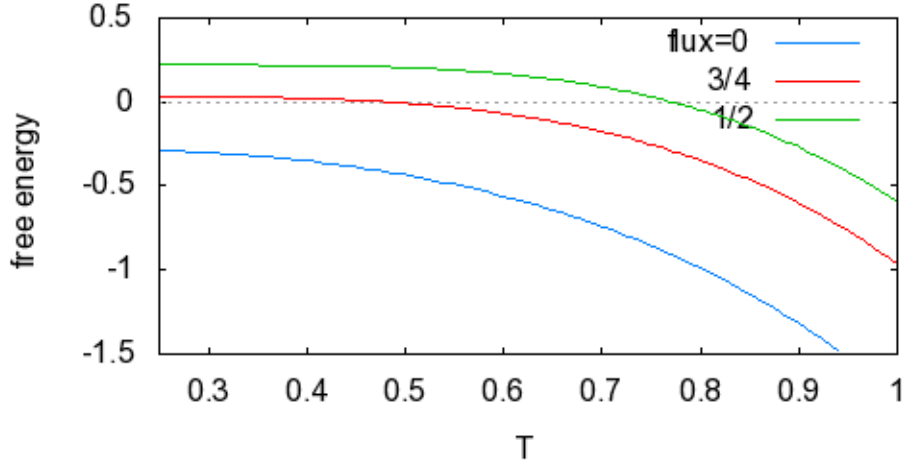
I have multiplied by two in (11) and (12) to allow for a complex field $\in U(1)$.

Everything in (11) is explicit and computable. In figure 4, I plot the free energy versus the temperature, $T = 1/\beta$, for an untwisted field for $q = 1, 2, 3$ and in figure 5 the free energy on the 4-lens for several twistings, *i.e.* fluxes. The high temperature behaviour, (10), can just about be seen.



⁶ The δ_0 in the numerator should read δ_0^2 .

Fig.5. 4-lens twisted free energy



7. Two-sided quotients

The same basic procedure can be applied to the inhomogeneous quotients, in the simplest case to the general lens spaces, $L(q; \lambda_1, \lambda_2)$. The only difference is that, when the degeneracy is being found, the necessary group average has to be done element by element. Without going into the standard details of the construction of the lens space, the elements of Γ' are determined by the angles $a\beta_1$ and β_2 ,

$$\frac{\beta_1}{2\pi} = \frac{p\nu_1}{q}, \quad \frac{\beta_2}{2\pi} = \frac{p\nu_2}{q}, \quad (13)$$

where $p, = 0, \dots, q-1$, labels γ . ν_1 and ν_2 are integers coprime to q , with λ_1 and λ_2 their mod q inverses.⁷

By an appropriate selection of a q -th root of unity, it is possible to set $\nu_1 = 1$, *i.e.* $\lambda_1 = 1$, without loss of generality. Any pair, (ν_1, ν_2) , can be reduced to $(1, \nu)$ by multiplying through by the mod q inverse of ν_1 .

The simple, one-sided lens space, $L(q; 1, 1)$, corresponds to setting $\nu = 1$ so that $\theta_L = 0$, $\theta_R = 2\pi p/q$.

⁷ Some authors label the lens space, equivalently, as $L(q; \nu_1, \nu_2)$.

The degeneracy generating function is, [1], ⁸

$$\begin{aligned} \sum_{l=0}^{\infty} d_l(q; \lambda_1, \lambda_2) t^l &= \frac{1}{q} \sum_{p=0}^{q-1} \sum_{l=0}^{\infty} \frac{\cos l\beta_1 - \cos l\beta_2}{\cos \beta_1 - \cos \beta_2} t^l \\ &= t(1 - t^2) \frac{1}{q} \sum_{p=0}^{q-1} \left(\frac{1}{1 + t^2 - 2t \cos \beta_1} \right) \left(\frac{1}{1 + t^2 - 2t \cos \beta_2} \right), \end{aligned} \quad (14)$$

with (13). The group sum over p has to be left until last.

Again putting $t = e^{-\tau}$, the 4d-conformal cylinder kernel this time is,

$$K^{1/2}(\tau; \lambda_1, \lambda_2) = \frac{\sinh \tau}{2q} \sum_{p=0}^{q-1} \frac{1}{(\cosh \tau - \cos \beta_1)(\cosh \tau - \cos \beta_2)}, \quad (15)$$

with (13). When $K^{1/2}(\tau)$ is antisymmetric in τ , as here and earlier,⁹ the familiar contour routine can be followed leading to the general expression for $Z'(0, \alpha)$,

$$Z'(0, \alpha) = \frac{1}{2} \int_0^{\infty} dx \operatorname{Re} \left(\frac{K^{1/2}(\tau)}{\tau} \cosh \alpha \tau \right), \quad \tau = x + i\Delta, \quad (16)$$

(of which (4) and (7) are examples). Δ lies between 0 and the imaginary part of the first pole of the cylinder kernel off the real τ axis.

As explained above, (13), it is sufficient to set $\nu_1 = 1$ and $\nu_2 = \nu$ with ν coprime to q . Then we have the corresponding $-\log \det$,

$$Z'_q(0, \alpha; \nu) = \frac{1}{q} \sum_{p=0}^{q-1} \int_0^{\infty} dx \operatorname{Re} \left(\frac{\sinh \tau \cosh \alpha \tau}{\tau (\cosh \tau - \cos \beta_1)(\cosh \tau - \cos \beta_2)} \right), \quad (17)$$

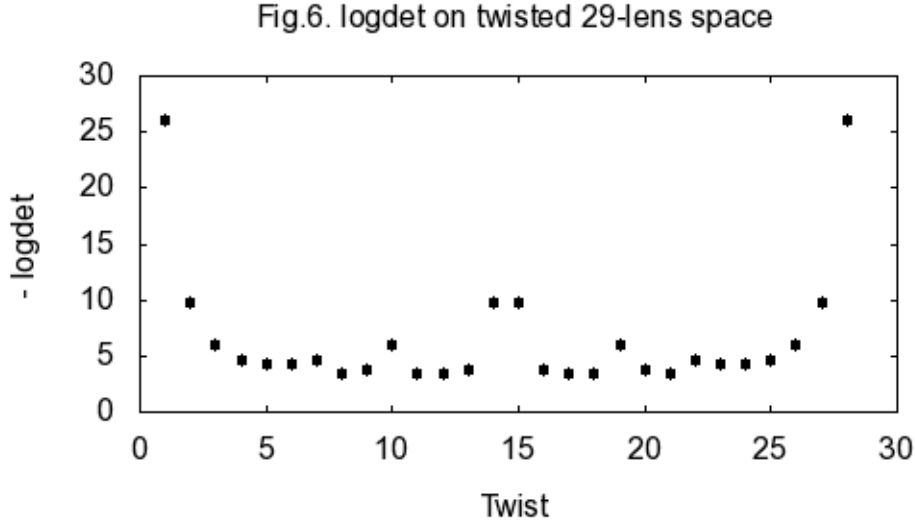
with $0 < \Delta < 2\pi/q$. This is easily computed.

It is convenient to make the cyclic order, q , an odd prime so that all ν from 1 to $q - 1$ are relevant. I refer to ν as the ‘twist’ of the manifold (as opposed to δ , the twist of the bundle which, in this section, is untwisted).

The logdet for $q = 29$ and $1 \leq \nu \leq 28$ is plotted in figure 6 for the 4d-conformal Laplacian. The effects of the isomorphisms between lens spaces can be seen. This is the same case figured in [2] computed by a different method so a comparison can be made. The results for the 3d-conformal Laplacian are very similar.

⁸ Higher dimensional spaces can be treated similarly *e.g.* [33,4]. A few remarks are in section 7.

⁹ The antisymmetry arises essentially from the harmonic factor, $(1 - t^2)$, in the generating function.



The *global* finite temperature considerations of section 6 can be directly extended to the inhomogeneous quotients. The Casimir energies would again be required to find the analogue of (11) for the total free energy. Analytical results for these are contained in [34] although, to be in keeping with the spirit of this paper, a simple numerical summation of the group averages would suffice.

8. Higher dimensions

The notion of lens space extends to higher odd-dimensional sphere quotients, S^{2e-1}/\mathbb{Z}_q (*e.g.* [35,9]). If $q = 2e - 1$ is an odd prime, they are catalogued by e integers, ν_i ($i = 1$ to e), the twists, coprime to q . The cylinder kernel for the Laplacian, conformal in $2e$ dimensions, is then,

$$K^{1/2}(\tau; \boldsymbol{\nu}) = \frac{\sinh \tau}{2q} \sum_{p=0}^{q-1} \prod_{i=1}^e \frac{1}{(\cosh \tau - \cos \beta_i)}, \quad (18)$$

where the angles β_i are given by,

$$\beta = \frac{2\pi \boldsymbol{\nu}}{q},$$

and the previous numerical procedure is effected without difficulty. If one just requires a number, this approach is easier than the more analytical routes leading to Hurwitz ζ -functions, say.

9. Minimal coupling

The value $\alpha = 1$ gives minimal coupling. There is then a zero mode ($l = 1$) which shows up as an infra-red divergence as τ tends to infinity in the relevant integrals such as (4). Removing the zero mode is conventional and amounts to subtracting $e^{-\tau}$ from the cylinder functions. This however destroys their antisymmetry and the contour method used earlier is impossible. Instead I employ a different approach which goes back to the early days of asymptotic analysis, see [36].

To begin with, I explicitly remove the (single) zero mode from the generating functions (cylinder kernels) by defining the subtracted functions,

$$\begin{aligned}\overline{K}^{1/2}(\tau) &= K^{1/2}(\tau) - e^{-\tau} \\ \overline{H}(\tau) &= H(\tau) - e^{-\tau} \\ \overline{Z}(s, \Gamma', \alpha) &= Z(s, \Gamma', \alpha) - \frac{1}{(1 - \alpha^2)^s},\end{aligned}\tag{19}$$

and note that $\overline{Z}(s, \Gamma', 1)$ is the minimal coupling ζ -function by definition.

Differentiating the basic ζ -function definition (1) *twice* with respect to α^2 yields a higher resolvent,

$$\left(\frac{d}{d\alpha^2}\right)^2 \overline{Z}(s, \Gamma', \alpha) = s(s+1) \sum_{l=2}^{\infty} \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^{s+2}}.\tag{20}$$

The sum converges at $s = 0$ and so

$$\left(\frac{d}{d\alpha^2}\right)^2 \overline{Z}'(0, \Gamma', \alpha) = \sum_{l=2}^{\infty} \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^2}.$$

The idea is to integrate this quantity twice, determining the constants of integration from the reference point $\alpha = 0$ since $\overline{Z}(s, \Gamma', 0) = Z(s, \Gamma', 0) - 1$, is known, [1].¹⁰

First one has,

$$\sum_{l=2}^{\infty} \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^2} = \sqrt{\pi} \int_0^{\infty} d\tau \overline{K}_{\Gamma'}^{1/2}(\tau) \left(\frac{\tau}{2\alpha}\right)^{3/2} I_{3/2}(\alpha\tau).\tag{21}$$

I now use the fact that, (*cf* (3)),

$$\overline{K}_{\Gamma'}^{1/2}(\tau) = -\frac{d}{d\tau} \overline{H}(\tau),$$

¹⁰ Any reference point could be chosen. This seems a convenient one, numerically.

to perform an integration by parts (the endpoint contributions vanish), which gives,

$$\begin{aligned}
\sum_{l=2}^{\infty} \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^2} &= \sqrt{\pi} \int_0^{\infty} d\tau \bar{H}(\tau) \frac{d}{d\tau} \left(\frac{\tau}{2\alpha} \right)^{3/2} I_{3/2}(\alpha\tau) \\
&= \sqrt{\pi} \int_0^{\infty} d\tau \bar{H}(\tau) \left(\frac{\tau}{2\alpha} \right)^{3/2} I_{1/2}(\alpha\tau) \\
&= \frac{1}{2} \int_0^{\infty} d\tau \tau \bar{H}(\tau) \frac{\sinh \alpha\tau}{\alpha}.
\end{aligned} \tag{22}$$

A first integration with respect to α^2 is now easy to perform, yielding

$$\frac{d}{d\alpha^2} \bar{Z}'(0, \Gamma', \alpha) = \int_0^{\infty} d\tau \bar{H}(\tau) (\cosh \alpha\tau - 1) + \frac{d}{d\alpha^2} \bar{Z}'(0, \Gamma', \alpha) \Big|_{\alpha=0}. \tag{23}$$

The constant of integration (the final term) is found from the standard relation,

$$\frac{d}{d\alpha^2} \bar{Z}(s, \Gamma', \alpha) = s \sum_{l=2}^{\infty} \frac{D_l(\Gamma')}{(l^2 - \alpha^2)^{s+1}} = s \bar{Z}(s+1, \Gamma', \alpha), \tag{24}$$

and one obtains for (23),

$$\begin{aligned}
\frac{d}{d\alpha^2} \bar{Z}'(0, \Gamma', \alpha) &= \int_0^{\infty} d\tau \bar{H}(\tau) (\cosh \alpha\tau - 1) + \bar{Z}(1, \Gamma', 0) \\
&= \int_0^{\infty} d\tau \bar{H}(\tau) (\cosh \alpha\tau - 1) + Z(1, \Gamma', 0) - 1,
\end{aligned}$$

using (19). I evaluate the constant later.

A final integration with respect to α^2 is required, and produces,

$$\begin{aligned}
\bar{Z}'(0, \Gamma', \alpha) &= \int_0^{\infty} \frac{d\tau}{\tau^2} \bar{H}(\tau) (2\alpha\tau \sinh \alpha\tau - 2 \cosh \alpha\tau + 2 - \alpha^2 \tau^2) + \\
&\quad \alpha^2 (Z(1, \Gamma', 0) - 1) + Z'(0, \Gamma', 0),
\end{aligned} \tag{25}$$

again using (19). The last term can be found numerically from the method in this paper, *e.g.* (4), as can the penultimate term. To be in harmony with the numerical viewpoint, the contour technique of [13] allows the values of $Z(n, \Gamma', \alpha)$ ($n \in \mathbb{Z}_+$) to be obtained easily. For example, I find, after partial integration, for $0 \leq \alpha < 1$,

$$Z(1, \Gamma', \alpha) = \int_0^{\infty} dx \operatorname{Re} (H(\tau) \cosh \alpha\tau), \quad \tau = x + i\Delta. \tag{26}$$

As before, it is sufficient to consider the cyclic quotient, $\Gamma' = \mathbb{Z}_q$, for which $H(\tau)$ is given in (3) and so ($\tau = x + i\Delta, 0 < \Delta < 4\pi/q$),

$$Z(1, q, \alpha) = \frac{1}{2} \int_0^{\infty} dx \operatorname{Re} \frac{\coth(q\tau/2) \cosh \alpha\tau}{\sinh \tau}. \tag{27}$$

Finally from (25), the minimal case is got by setting $\alpha = 1$ to give my computational formula for the minimal $-\log\det$ on homogeneous lens spaces,

$$\begin{aligned} \overline{Z}'(0, q, 1) = \int_0^\infty \frac{d\tau}{\tau^2} \left(\frac{\coth(q\tau/2)}{2 \sinh \tau} - e^{-\tau} \right) (2\tau \sinh \tau - 2 \cosh \tau + 2 - \tau^2) \\ + Z(1, q, 0) - 1 + Z'(0, q, 0). \end{aligned} \quad (28)$$

One sees that the method of back integration has produced a systematic method of subtraction that simultaneously tempers the UV and IR divergences.

The value $\alpha = 0$ gives the 4d-conformal operator and, although not needed, closed form expressions for the relevant quantities can be found in [1]. For example, for even-ordered lens spaces, one has the simple result,

$$Z(1, 2q, 0) = \frac{\pi}{4q} \sum_{p=1}^{q-1} \operatorname{cosec}(\pi p/q). \quad (29)$$

This agrees numerically with (27) from which, of course, it can be derived.

The case $q = 2$ provides a simple check. The quotient manifold is projective three-space, P^3 . Periodicity restricts l to being odd, [17], and the ζ -function rewrites to the difference of those on the three-sphere for minimal, 'm', ($\alpha = 1$) and 3d-conformal, 'c', ($\alpha = 1/2$) propagation. Precisely,

$$\zeta_{P^3}^m(s) = \zeta_{S^3}^m(s) - 2^{2-2s} \zeta_{S^3}^c(s),$$

so that,

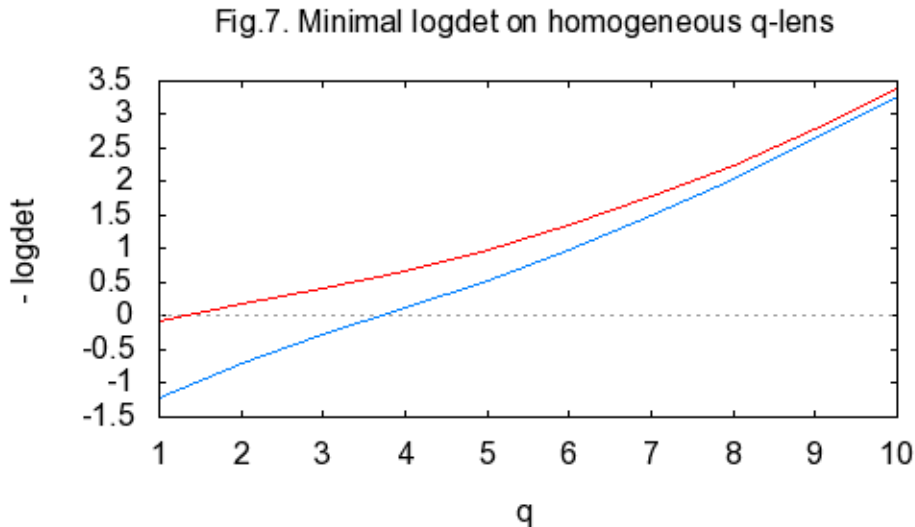
$$\zeta_{P^3}^{m'}(0) = \zeta_{S^3}^{m'}(0) - 4 \zeta_{S^3}^{c'}(0), \quad (30)$$

where I have used the fact that $\zeta_{S^3}^c(0)$ vanishes, reflecting the closed nature of S^3 and the absence of a zero mode.

Both values on the right-hand side of (30) for the full sphere were evaluated analytically a long time ago giving $\zeta_{P^3}^{m'}(0) = -0.695171$, agreeing with the quadrature, (28).¹¹

There do not seem to be many published *numerical* values for lens space determinants. Nash and O'Connor, [3], give some quite involved expressions with their corresponding plots, which seem to agree with my results. Figure 7 below gives an indication of my values, which, as an aside, suggest an inflection. For comparison, the upper curve is for the 4d conformal Laplacian (see fig.1), *i.e.* the last term in (28).

¹¹Simplifications arise for this case. For example $Z(1, 2, 0)$ vanishes.



Using $\alpha = 0$ as a reference point is related to the method of binomially expanding the ζ -function, (1), in powers of α^2 , utilised in [37]; see [38].

10. Discussion

Relatively simple expressions allow the determinants on any homogeneous quotient of the three-sphere to be computed for any mass and for any twisting by an internal symmetry group. The inhomogeneous case has been treated in less detail.

I have given some examples and plotted a few graphs, the physical significance of which is somewhat moot, and are meant only to illustrate possibilities. It is hoped that the numerical approach will be useful in confirming analytical results.

I have discussed only scalar fields, but the extensions to spins half and one are straightforward, *cf* [1].

A different technique has been used for minimal coupling which involves back integrating the expression for a higher resolvent. This method could also be used for the other couplings although then the reference $\alpha = 0$ values would need to be found independently.

Another set of quotients that lend themselves readily to similar treatment are the flat ones, classified by Hantsche and Wendt, on which a certain amount of work has been done. I draw attention now only to the early analysis of Unwin, [31,39], on the finite temperature field theory and resulting thermodynamics. Earlier, zero

temperature calculations can be found in [17,40] and de Witt, Hart and Isham, [41].

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